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A Two-Dimensional Model of Reaction-Diffusion System as a Generator of Old Hebrew Letters

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Minimal models for many patterns observed in nature should be based on systems, in which only chemical reactions and diffusion transport occur (reaction-diffusion systems). In order to present a richness of patterns possible in such systems, we show here the asymptotic solutions to the nonlinear, partial, parabolic equations with zero flux boundary conditions in the form of patterns imitating all Old Hebrew letters (the Siloam inscription) obtained in two-dimensional systems. All letters are obtained in the same model, but sizes of the systems and initial conditions are different for each letter. The chemical model consists only of elementary reactions.

Key words: nonlinear reaction-diffusion systems, patterns

Minimal models of spatio-temporal patterns observed in natural systems can be based on nonlinear reaction-diffusion systems [1–11]. This assumption is supported by experimental observations of various patterns in real chemical systems, in which transport processes are limited to diffusion. Chemical waves, target patterns (traveling concentric rings) [12], spiral waves [13], stationary periodical structures (the large amplitude Turing structures) [14,15], lamellar structures [16], oscillating waves [17], self-replicating spots [18] and others [19] have been observed in chemical systems like the Belousov-Zhabotinsky reaction [2], the CIMA system (chloride - iodide - malonic acid) [9–11] and others. Patterns observed in nature have been used as norms for designs invented by people in arts and sciences. For example the letters in the Old Hebrew Alphabet (the Siloam inscription) have been imitated on shapes observed in nature [20].

In the present paper we illustrate the creation of large amplitude stationary patterns in a reaction-diffusion system, which have the form of all Old Hebrew letters (the Siloam inscription). We want to stress, that all letters have been created using the same reaction-diffusion model. Each letter has been obtained in the 2D system with appropriate sizes as the result of local excitations of a homogeneous stationary state, which is stable to small perturbations. Unlike from the all capital Latin letters [21], in which all systems have been convex ones, in the case of some Old Hebrew letters we have to use concave systems. In our case the convex areas are rectangles or squars. Each two points of such areas can be joint by a straight interval belonging totally to them. The concave areas are rectangular polygons with numbers of apexes equal to 6, 8 or 10. They have points, which cannot be joint by a straight interval belonging totally to them.

The model consists of the following elementary, mono-molecular and bi-molecular reactions (excluding autocatalysis) [21,22]:

$$S_0 \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} S \quad (1) \qquad S + E \underset{k_{-2}}{\overset{k_2}{\rightleftarrows}} SE \quad (2) \qquad SE \xrightarrow{k_3} E + P \quad (3)$$

$$SE + S \underset{k_{-4}}{\overset{k_4}{\rightleftarrows}} S_2E \quad (4) \qquad P + E \underset{k_{-5}}{\overset{k_5}{\rightleftarrows}} EP \quad (5) \qquad P + SE \underset{k_{-5}}{\overset{k_5}{\rightleftarrows}} SEP \quad (6)$$

$$P + S_2E \underset{k_{-5}}{\overset{k_5}{\rightleftarrows}} S_2EP \; (7) \quad P + E' \underset{k_{-6}}{\overset{k_6}{\rightleftarrows}} E'P \; (8) \quad E'P \xrightarrow{k_7} E' + R \; (9) \quad P \xrightarrow{\mathrm{k}_8} Q \quad (10)$$

The reactant S_0 is treated as the reservoir variable, whose concentration is maintained constant. A catalytic (enzymatic) reaction is described by the steps (2)–(7). This reaction is inhibited by an excess of the reactant S and the product P. For simplicity we assume that rate constants in steps (5)–(7) are the same. This is reasonable assumption for allosteric inhibition by the product. In the steps (8) and (9) the product P is consumed by another enzymatic reaction with the enzyme E' producing inreactive product R. We assume that this reaction occur in its saturation regime, what allows on simplification of formulas for a nullcline for the product. In the step (10) P is transformed directly to the product Q. The scheme describes an open chemical system due to the steps (1) and (8)–(10).

In experimental systems total concentrations of catalysts (enzymes) $[E_0]$ and $[E_0^{'}]$ are much smaller than concentrations of reactants and products. Due to this assumption, it is possible to separate of time scales, in which the concentrations of the reagents change. All concentrations of the catalyst (enzymes) and their complexes become fast variables, whereas S and P are the slow ones. In slow time scale, the fast variables are equal to their quasistationary values [23], what means that changes of their concentrations in this time scale are equal to zero. In the slow time scale the dynamics of the system can be described by a reduced system of slow variables. Neglecting diffusion of the catalysts (enzymes) and all their complexes, the dynamics of inhomogeneous 2D system can be described by the following reaction-diffusion equations:

$$\frac{\partial s}{\partial t} - D_s \frac{\partial^2 s}{\partial x^2} - D_s \frac{\partial^2 s}{\partial y^2} = A_1 - A_2 s - \frac{s}{(1 + s + A_3 s^2)(1 + p)} \tag{11}$$

$$\frac{\partial p}{\partial t} - D_p \frac{\partial^2 p}{\partial x^2} - D_p \frac{\partial^2 p}{\partial y^2} = B\left(-B_1 - B_2 p + \frac{s}{(1 + s + A_3 s^2)(1 + p)}\right) \tag{12}$$

where: $s = \frac{[S]}{K_m}$ and $p = K_5[P]$, are dimensionless concentrations of S and P respectively, $t = \frac{k_3[E_0]}{K_m} t'$ is dimensionless time (t' is real time), x and y are dimensionless space variables, and D_s and D_p are dimensionless diffusion coefficients. The parameters are defined as follows:

$$A_1 = \frac{k_1[S_0]}{k_3[E_0]},\; A_2 = \frac{k_{-1}K_m}{k_3[E_0]},\; A_3 = \frac{k_4}{k_{-4}}K_m,\; B = K_m K_5,\; B_1 = \frac{k_7[E_0']}{k_3[E_0]},\; B_2 = \frac{k_8}{k_3[E_0]K_5},$$

where $K_m = \frac{k_{-2} + k_3}{k_2}$, $K_5 = \frac{k_5}{k_{-5}}$.

The assumption that the second catalytic reaction occurs in a saturation regime ($K_5(k_7 + k_{-6}) << k_6 p$) allows us to describe the consumption of p by the reactions (8)–(9) by the parameter B_1.

We consider the initial-boundary value (Fourier) problem with zero-flux (Neumann) boundary conditions:

$$\frac{\partial s}{\partial x} = \frac{\partial s}{\partial y} = \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \text{ for boundaries of the system.}$$

Let us notice, that (11–12) are symmetrical with respect to translations and reflections in x and y. Therefore, if $s(t,x,y)$ is a solution to the system, then its mirror reflection is also the solution. Because of the mirror symmetry for the zero-flux boundary conditions, any solution together with its mirror reflection is also the solution on the enlarged region. These properties allow to compose some patterns from their parts by mirror reflections along to one or both space coordinates and moreover, to safe computer time.

We assume the same values of the parameters as in our previous papers [21,22]. In all calculations we assume: $A_1 = 0.01$, $A_2 = 10^{-4}$, $A_3 = 0.505$, $B = 0.625$, $B_1 = 7.99 \times 10^{-3}$ and $B_2 = 4.65 \times 10^{-5}$ and the diffusion coefficients: $D_s = 10^{-5}$ and $D_p = 5 \times 10^{-5}$. At these values of the parameters, the homogeneous system is in an excitable regime with three stationary states. One of them is a stable node and the other ones are a saddle-point and an unstable focus [22]. Our recent results shown, that initial local excitations of 1D system with above values of the parameters evolve to the large amplitude stationary periodical structures [22]. Each stationary periodical structure exists on a limited interval in 1D space. On each interval, which is longer than the critical one, some number of stationary periodical structures exists. These patterns consist of subsequent numbers of half-periods. The number of coexisting structures substantially increases with the size of the system. The coexistence of the large amplitude stationary structures exists also in 2D systems, and this property is the crucial one, which allows us to construct the patterns with desired shapes.

In order to generate the appropriate patterns, we have changed sizes of the system and positions of local excitations of the homogeneous stable stationary state. In all regions, which are initially excited we assume $s(0,x,y) = 20$ and $p(0,x,y) = 35$. Outside of excited regions, we assume initial distributions of s and p equal to the homogeneous stationary values for both variables.

In generation of the patterns, corresponding to all Old Hebrew letters, we have to use convex areas as well as concave ones. In nature both types of shapes of systems exist. All systems presented in this paper have the form of rectangular polygons with corresponding number of apexes. All letters shown in Fig. 1 have been generated for the data presented in Table 1. N_i denotes the number of apexes in the rectangular polygon, which defines the area of the system. x_i and y_i denote positions of the apexes of the corresponding polygon on the (x,y) plane. The number rectangular regions excited is denoted by n. Positions of the lower left apex of excited rectangles are denoted by X_n and Y_n and l_{X_n}, and l_{Y_n} denote the sizes of excited rectangles in x and y directions, respectively.

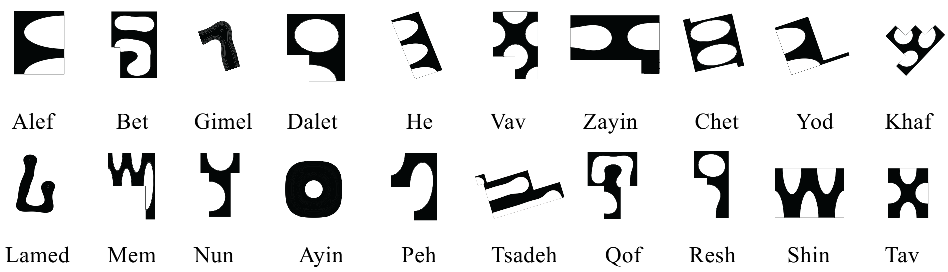

Figure 1. The set of asymptotic patterns generated in twenty 2D systems. The patterns have been obtained in convex or concave systems with sizes and initial conditions shown in Table 1. The patterns are little deformed to give the letters with close height and width. All letters are formed from the patterns by separations of asymptotic values of $s(t,x,y)$ into two regions with values of s higher or lower than 2. One of these two regions is marked in black.

If maximal local changes in $s(t,x,y)$ and $p(t,x,y)$ appear at most 8-th digit and continue the decreasing in time, we have assumed that the pattern has an asymptotic form. The asymptotic values of $s(t,x,y)$ and $p(t,x,y)$ change in the ranges (0.1–45) and (8–40), respectively. The letters shown in Fig. 1 are formed from the asymptotic patterns by separation of asymptotic values of $s(t,x,y)$ into two regions. The region in the (x,y) plane is marked in black where $s(t,x,y)$ is higher than 2 only for the letter Ayin. For the all other letters the black areas mean the regions where $s(t,x,y)$ lower than 2.

The patterns obtained in the calculations have been squeezed or/and elongated in one or two dimensions, in order to get the letters with proper height and width.

Some of the letters are rather scribble, but they seem to be readable. Main visual deformations are caused by the fact, that they are generated in systems composed from rectangular polygons. This is the reason, that some of them have sharp corners. In order to avoid such deformations, one should replace the rectangular systems by ones with smooth boundaries.

Table 1. Number of apexes of rectangular polygons (N_i), their positions (x_i,y_i), number of excited regions (n), their positions (X_n, Y_n) and sizes (l_{X_n},l_{Y_n}).

Letter	*N*	(x_i,y_i)	*n*	(X_n, Y_n)	(l_{X_n},l_{Y_n})
Alef	4	(0.0,0.0); (0.0,3.0); (1.0,3.0); (1.0,0.0)	2	(0.76,0.0); (0.76,1.72)	(0.24,0.24); (0.48,0.48)
Bet	6	(0.8,0.0); (0.8,3.4); (0.0,3.4); (0.0,7.4); (4.0,7.4); (4.0,0.0)	2	(2.24,3.8); (1.76,5.16)	(0.24,0.24); (0.44,0.44)
Gimel	6	(0.4,0.0); (0.4,1.52); (0.0,1.52); (0.0,4.96); (3.44,4.96); (3.44,0.0)	2	(1.6,3.12); (1.88,0.0)	(0.24,0.24); (0.48,0.24)
Dalet	6	(1.0,0.0); (1.0,1.6); (0.0,1.6); (0.0,4.0); (2.6,4.0); (2.6,0.0)	2	(0.96,2.76); (1.0,0.0)	(0.48,0.48); (0.24,0.48)
He	4	(0.0,0.0); (0.0,5.0); (1.0,5.0); (0.0,5.0)	3	(0.0,0.0); (0.0,1.78); (0.0,3.78)	(0.24,0.24); (0.24,0.48); (0.24,0.48)
Vav	6	(1.72,0.0); (1.72,1.72); (0.0,1.72); (0.0,5.16); (3.44,5.16); (3.44,0.0)	5	(0.0,3.2); (1.48,4.92); (3.2,3.2); (3.2,0.0); (1.72,1.72)	(0.24,0.28); (0.48,0.24); (0.24,0.48); (0.24,0.24); (0.24,0.24)
Zayin	6	(3.6,0.0); (3.6,0.4); (0.0,0.4); (0.0,3.2); (4.0,3.2); (4.0,0.0)	1	(1.4,1.68)	(1.2,0.24)
Chet	8	(1.72,0.0); (1.72,0.4); (0.0,0.4); (0.0,4.8); (0.28,4.8); (0.28,4.4); (2.0,4.4); (2.0,0.0)	2	(0.88,1.28); (0.88,3.28)	(0.24,0.24); (0.24,0.24)
Yod	6	(0.0,0.0); (0.0,3.0); (1.0,3.0); (1.0,0.2); (1.8,0.2); (1.8,0.0)	2	(0.0,0.0); (0.0,1.72)	(0.24,0.24); (0.24,0.48)
Kaf, Khaf	10	(0.0,0.0); (0.0,1.72); (1.72,1.72); (1.72,5.16); (3.44,5.16); (3.44,3.44); (5.16,3.44); (5.16.1.72); (6.88,1.72); (6.88,0.0)	1	(3.22,1.50)	(0.48,0.48)
Lamed	8	(0.0,0.0); (0.0,6.0); (0.4,6.0); (0.4,10.0); (3.4,10.0); (3.4,6.0); (6.0,6.0); (6.0,0.0)	1	(2.76,2.76)	(0.48,0.48)
Mem	6	(6.88,0.0); (6.88,1.72); (0.0,1.72); (0.0,3.44); (8.6,3.44); (8.6,0.0)	3	(0.0,3.32); (3.3,3.32); (6.74,3.32)	(0.12,0.12); (0.24,0.12); (0.24,0.12)
Nun	8	(0.68,0.0); (0.68,3.4); (0.0,3.4); (0.0,4.92); (3.04,4.92); (3.04,3.4); (2.76,3.4); (2.76.0.04)	2	(1.28,4.68); (0.68,1.56)	(0.48,0.24); (0.24,0.12)
Ayin	4	(0.0,0.0); (0.0,4.0); (4.0,4.0); (4.0,0.0)	1	(1.88,1.88)	(0.24,0.24)
Peh, Feh	6	(1.72,0.0); (1.72,1.72); (0.0,1.72); (0.0,3.44); (3.44,3.44); (3.44,0.0)	1	(1.48,3.2)	(0.48,0.24)

Table 1 (continuation)

Tsadeh	10	(0.64,0.0); (0.64,4.0); (0.0,4.0); (0.0,5.0); (1.0,5.0); (1.0,4.0); (5.64,4.0); (5.64,1.75); (8.64,1.75); (8.64,0.0)	3	(0.64,0.0); (0.64,2.16); (0.0,4.0)	(6.4,0.48); (3.6,0.48); (0.48,0.48)
Qof	8	(2.0,0.0); (2.0,4.0); (0.0,4.0); (0.0,8.0); (6.0,8.0); (6.0,4.0); (4.0,4.0); (4.0,0.0)	1	(1.78,4.04)	(0.24,0.24)
Resh	6	(1.0,0.0); (1.0,3.6); (0.0,3.6); (0.0,6.0); (2.6,6.0); (2.6,0.0)	2	(1.0,1.34); (0.96,4.76)	(0.24,0.88); (0.48,0.48)
Shin	4	(0.0,0.0); (0.0,1.72); (6.88,1.72); (6.88,0.0)	3	(0.0,1.6); (3.32,1.6); (6.76,1.6)	(0.12,0.12); (0.24,0.12); (0.12,0.12)
Tav	4	(0.0,0.0); (0.0,3.44); (3.44,3.44); (3.44,0.0)	1	(1.6,1.6)	(0.24,0.24)

Some letters presented in Fig. 1 may be generated in a few ways, that means on polygons with sizes different from these shown in Table 1 and local excitations positioned in various places. Let us mention, that some systems shown in Table 1 are symmetrical along one or both coordinates. It is obvious, that an initial, local excitation positioned symmetrically in these 2D systems evolves according to their own symmetry and asymptotically gives a symmetrical pattern. In such systems an unsymmetrical initial disturbance with small asymmetry evolves asymptotically to a symmetric pattern, whereas that with strong asymmetry gives sometimes unsymmetrical asymptotic patterns.

Small changes in sizes of the convex and concave systems, which do not disturb their symmetries, as well as small changes in positions of initial excitations usually give small differences in calculated values of $s(t,x,y)$ and do not change asymptotic shapes of the pattern. However, if one size of the system is close to the interval, on which 1D pattern becomes unstable, then in this case small changes of the sizes may give completely different asymptotic patterns. The model is structurally stable, what means that there are such sufficiently small changes of the right hand sides of the equations, as well as the diffusion coefficients, which do not cause qualitative differences in asymptotic patterns.

The asymptotic patterns presented in this paper, as well as those described in our previous one [21], are initiated by the given local excitations. In excitable systems inhomogeneities may appear due to internal, local fluctuations. There is higher from zero probability that such fluctuations can induce the spontaneous formation of the large amplitude patterns with the shapes presented in the paper. The patterns in the form of all Old Hebrew letters may appear in various dynamical systems, provided necessary conditions are fulfilled. The coexistence of stationary patterns, that is the dependence of asymptotic solutions on initial conditions is the crucial point. This coexistence is achieved, due to a sufficiently large difference in the diffusion coefficients in two-variable systems. For equal or very close diffusion coefficients an initial, local excitation evolves always to the travelling impulse, that is to the pulse of excitation running through the system with a constant velocity. The patterns presented above should be considered only as examples of the variety of asymptotic structures possible in nonlinear, excitable chemical systems. They support the opinion, that reaction-diffusion systems can be useful models for the generation of patterns in biological systems in the process of cell differentiation, which is governed by the positional information [24]. Let us notice, that the possibility to create patterns with desired forms open new prospects for encoding information in chemical systems.

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